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Wronskian and rational solutions of the differential–difference KP equation

Thamizharasi Tamizhmani†§, S Kanaga Vel‡ and K M Tamizhmani‡

† Department of Mathematics, K M Centre for Post Graduate Studies, Pondicherry-605 008, India

‡ Department of Mathematics, Pondicherry University, Kalapet, Pondicherry-605 014, India

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Abstract. We present the Wronskian form of the N-soliton solutions of the differential–difference Kadomtsev–Petviashvilli (DΔKP) equation. Also a wider class of rational solutions are derived using semi-discrete analogue of Schur polynomials and its generalization. Our approach is based on Sato theory formalism which gives all these solutions naturally.

1. Introduction

It is well known that the soliton equations can be rewritten in bilinear form through a suitable dependent variable transformation. The solution of this bilinear form can be obtained either through perturbation or the Wronskian technique [1–5]. This is popularly known as Hirota's bilinear method [1]. Using this technique, Hietarinta systematically classified Hirota's bilinear forms which possess three-Soliton solutions [2]. It is one of the direct methods widely used because of its simplicity and efficiency. In the usual Hirota's bilinear method, one uses perturbational expansions to the bilinear equations to obtain the soliton solutions in exponential form. The compact form of expressing the N-soliton solutions for Hirota's bilinear equations was first introduced by Satsuma [3] and further developed by Freeman and Nimmo [4] in terms of the Wronskian determinant. This procedure has been applied to the KP [5], the Boussinesq [6], and other soliton systems [7–9]. It is also known that the N-soliton solutions of KP hierarchy can be derived through Sato theory [10–12], which is expressed by the τ function [13, 14]. This τ function can be expressed in the form of the generalized Wronskian determinant defined on the infinite-dimensional Grassmannian manifold [10–14]. In this framework, Hirota's bilinear forms arise naturally as Plücker relations. Hence, it is convenient to deal with the Wronskian determinant in order to find the N-soliton solutions. Using the Laplace expansion of the determinant, we can easily verify that the τ function satisfies the given Hirota's bilinear form. It is remarkable that through this representation the algebraic properties and the structure of the solutions can be explained explicitly [15–18].

It has been recognized that integrable systems, in the sense of inverse scattering transform (IST) [19], possess other classes of solutions as well, called rational solutions. Deriving rational solutions for soliton systems is very important and they often play a significant role in explaining the physical applications of the system. The class of rational

§ E-mail address: tamizh@yahoo.com

solutions for the KdV equation was first investigated by Airault *et al* [20]. Consequently, Ablowitz and Satsuma [21] proposed a direct approach to study the rational solutions of soliton systems, by taking the long-wave limit in the multisoliton solutions. Nimmo and Freeman [22] represented these rational solutions in the Wronskian determinant form. Since then, various methods have been proposed for obtaining rational solutions for integrable systems, notably by Airault and Moser [23], Adler and Moser [24], Satsuma and Ishimori [25], Gilson and Nimmo [26], Nakamura [27], Pelinovsky [28] and Hu [29]. Usually, one can obtain the higher rational solutions through the repeated application of Bäcklund transformations on the known, simple, rational solution. On the other hand, as already mentioned, Sato theory provides a systematic approach to find the rational solutions of KP hierarchy. The fundamental ones are represented in terms of Schur polynomials which satisfy a certain set of linear differential equations. Thus rational solutions are special examples of the general Wronskian solutions obtained through Sato theory. Although intensive research is going on in this direction for continuous integrable systems, the search for the rational solutions for differential–difference equations has been relatively low-key [30]. In fact it is other aspects of differential–difference equations that have attracted most attention in recent years [31–49].

Using the differential–difference analogue of Sato theory [49], we derived Lax pairs, symmetries and conservation laws of the DΔKP equation. In this paper, we employ the bilinear formalism and present the N-soliton solutions and also give explicitly the rational solutions of the $D\Delta$KP equation. We observe that rational solutions are merely the consequence of specializing the original τ determinant derived for N-soliton solutions, using Sato theory.

2. Wronskian solution

Sato introduced a powerful tool by which it has been shown that a hierarchy of equations can be derived, all of which have common solutions expressed in terms of the τ function involving an infinite number of independent variables. Here we use Sato theory [10–12, 49] framework to derive the N-soliton solutions in terms of the Wronskian for the DΔKP equation.

Now, we consider the DΔKP equation in the form

$$\Delta\left(\frac{\partial u}{\partial t_2}+2\frac{\partial u}{\partial t_1}-2u\frac{\partial u}{\partial t_1}\right)=(2+\Delta)\frac{\partial^2 u}{\partial t_1^2} \tag{1}$$

where $u=u(t_1,t_2,n)$ and Δ denotes the forward difference operator defined by $\Delta f(n)=f(n+1)-f(n)$ and the shift operator E defined by $Ef(n)=f(n+1)$. The operators Δ and E are connected by $\Delta=E-1$. This equation was first derived by Date *et al* [48] and has been the subject of more detailed study recently [49]. Equation (1) can be written in bilinear form through the variable transformation given by

$$u(t_1,t_2,n)=\frac{\partial}{\partial t_1}\log\frac{\tau_{n+1}}{\tau_n}. \tag{2}$$

Substituting this transformation in equation (1) we get the bilinear form

$$(D_{t_2}+2D_{t_1}-D_{t_1}^2)\tau_{n+1}\cdot\tau_n=0. \tag{3}$$

Here D denotes the Hirota bilinear operator [1, 2]. We first expand equation (3) using the

Hirota operator, and get

$$\tau_n \frac{\partial \tau_{n+1}}{\partial t_2} - \tau_{n+1}\frac{\partial \tau_n}{\partial t_2} + 2\tau_n \frac{\partial \tau_{n+1}}{\partial t_1} - 2\tau_{n+1}\frac{\partial \tau_n}{\partial t_1} + 2\frac{\partial \tau_{n+1}}{\partial t_1}\frac{\partial \tau_n}{\partial t_1} - \tau_n \frac{\partial^2 \tau_{n+1}}{\partial t_1^2} - \tau_{n+1}\frac{\partial^2 \tau_n}{\partial t_1^2} = 0. \tag{4}$$

We prove in the following that the solution of the D∆KP equation can be written in the compact form using the Wronskian (Casorati) determinant

$$\begin{aligned} \tau_n &= W(f_n^{(1)}, f_n^{(2)}, \ldots, f_n^{(N)}) \\ W &= \begin{vmatrix} f_n^{(1)} & \Delta f_n^{(1)} & \cdots & \Delta^{N-1} f_n^{(1)} \\ f_n^{(2)} & \Delta f_n^{(2)} & \cdots & \Delta^{N-1} f_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_n^{(N)} & \Delta f_n^{(N)} & \cdots & \Delta^{N-1} f_n^{(N)} \end{vmatrix} \\ &= \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdots & f_{n+N-1}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix} \end{aligned} \tag{5}$$

where $f_n^{(j)} = f^{(j)}(t_1, t_2, n)$, $j = 1, 2, \ldots, N$ are the solutions of a set of linear partial differential–difference equations given by

$$\frac{\partial f_n^{(j)}}{\partial t_1} = \Delta f_n^{(j)} \qquad \frac{\partial f_n^{(j)}}{\partial t_2} = \Delta^2 f_n^{(j)} \qquad j = 1, 2, \ldots, N. \tag{6}$$

A particular solution of (6) is readily given by

$$f_n^{(j)} = (1 + p_j)^n \exp(p_j t_1 + p_j^2 t_2) \qquad j = 1, 2, \ldots, N. \tag{7}$$

In order to construct the N-soliton solutions, we take the form of $f_n^{(j)}$ as

$$f_n^{(j)} = \exp \eta_j + \exp \xi_j \tag{8}$$

with

$$\eta_j = p_j t_1 + p_j^2 t_2 + n \log(1 + p_j) + \eta_{j0} \tag{9}$$

and

$$\xi_j = q_j t_1 + q_j^2 t_2 + n \log(1 + q_j) + \xi_{j0}. \tag{10}$$

Following Nimmo and Freeman's notation [4, 5] let us denote τ_n in (5) as

$$\tau_n = |0, 1, 2, \ldots, N-1| = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdots & f_{n+N-1}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix}. \tag{11}$$

The terms involved in (4) may then easily be computed as

$$
\begin{aligned}
\tau_{n+1} &= |1,2,\ldots,N| \\
\frac{\partial\tau_n}{\partial t_1} &= |0,1,2,\ldots,N-2,N| - N|0,1,2,\ldots,N-1| \\
\frac{\partial\tau_{n+1}}{\partial t_1} &= |1,2,3,\ldots,N-1,N+1| - N|1,2,\ldots,N| \\
\frac{\partial\tau_n}{\partial t_2} &= N|0,1,2,\ldots,N-1| - |0,1,2,\ldots,N-3,N-1,N| \\
&\quad + |0,1,2,\ldots,N-2,N+1| - 2|0,1,2,\ldots,N-2,N| \\
\frac{\partial\tau_{n+1}}{\partial t_2} &= N|1,2,\ldots,N| - |1,2,\ldots,N-2,N,N+1| \\
&\quad + |1,2,\ldots,N-1,N+2| - 2|1,2,\ldots,N-1,N+1| \\
\frac{\partial^2\tau_n}{\partial {t_1}^2} &= N^2|0,1,2,\ldots,N-1| - 2N|0,1,2,\ldots,N-2,N| \\
&\quad + |0,1,2,\ldots,N-3,N-1,N| + |0,1,2,\ldots,N-2,N+1| \\
\frac{\partial^2\tau_{n+1}}{\partial {t_1}^2} &= N^2|1,2,\ldots,N| - 2N|1,2,\ldots,N-1,N+1| \\
&\quad + |1,2,\ldots,N-2,N,N+1| + |1,2,\ldots,N-1,N+2|
\end{aligned}
\tag{12}
$$

where the linear equations (6) are used in the derivation of the above results. Using the above expressions, we see that the left-hand side of equation (4) reduces to

$$
\begin{aligned}
&2|0,1,2,\ldots,N-1||1,2,\ldots,N-2,N,N+1| \\
&-2|0,1,2,\ldots,N-2,N||1,2,\ldots,N-1,N+1| \\
&+2|0,1,2,\ldots,N-2,N+1||1,2,\ldots,N|
\end{aligned}
$$

which is the Laplace expansion of the $2N \times 2N$ determinant

$$
\begin{vmatrix}
0 & \widehat{N-2} & \bigcirc & N-1 & N & N+1 \\
0 & \bigcirc & \widehat{N-2} & N-1 & N & N+1
\end{vmatrix}
\tag{13}
$$

where $\widehat{N-2} = |1,2,\ldots,N-2|$ and $\bigcirc$ denotes the $(N-2)\times(N-2)$ zero matrix. Since the above determinant is zero it indeed verifies that τ_n satisfies the bilinear equation (4) identically.

3. Rational solutions

In this section, we describe the method of finding the class of rational solutions for the DΔKP equation and show that they are merely a particular case of the N-soliton solutions given in terms of the Wronskian (5). For this purpose, we consider the set of linear partial differential–difference equations (6) with (7) as particular solution. Note that $f_n^{(j)}$ in (7) can be expressed as a power series in p_j and hence we have

$$
(1+p_j)^n \exp(p_j t_1 + p_j^2 t_2) = \sum_{m=0}^{\infty} P_m p_j^m.
$$

Expanding the left-hand side of the above equation and comparing the like powers of p_j on both sides we arrive at the semi-discrete analogue of the Schur polynomials and they can be expressed in a compact way as

$$P_m = \sum_{\substack{\alpha_0,\alpha_1,\alpha_2\geqslant 0\\ \alpha_0+\alpha_1+2\alpha_2=m}} \frac{n(n-1)(n-2)\ldots(n-\alpha_0+1)t_1^{\alpha_1}t_2^{\alpha_2}}{\alpha_0!\alpha_1!\alpha_2!} \tag{14}$$

where $P_m = 0$, $\forall m \leqslant 0$, $\Delta P_m = P_{m-1}$, $\partial P_m/\partial t_1 = \Delta P_m$, and $\partial P_m/\partial t_2 = \Delta^2 P_m$. They indeed satisfy the equation (3). On using (2), they become rational solutions of (1). It is interesting to note that these Schur polynomials can be used to generate a further class of rational solutions given by

$$P_{l_1 l_2\cdots l_N} = \begin{vmatrix} P_{l_1} & P_{l_2} & \cdots & P_{l_N} \\ P_{l_1-1} & P_{l_2-1} & \cdots & P_{l_N-1} \\ \vdots & \vdots & \ddots & \vdots \\ P_{l_1-N+1} & P_{l_2-N+1} & \cdots & P_{l_N-N+1} \end{vmatrix} \tag{15}$$

where $l_1, l_2, \ldots, l_N$ are distinct integers. We list below the first few rational solutions generated using (14) and (15):

$$\begin{aligned}
P_0 &= 1\\
P_1 &= n + t_1\\
P_2 &= \frac{n(n-1)}{2!} + \frac{t_1^2}{2!} + nt_1 + t_2\\
P_3 &= \frac{n(n-1)(n-2)}{3!} + \frac{t_1^3}{3!} + \frac{n(n-1)}{2!}t_1 + \frac{nt_1^2}{2!} + nt_2 + t_1t_2\\
P_{12} &= \frac{n}{2} + \frac{n^2}{2} - t_2 + nt_1 + \frac{t_1^2}{2}\\
P_{13} &= \frac{-n}{3} + \frac{n^3}{3} + n^2t_1 + nt_1^2 + \frac{t_1^3}{3}\\
P_{23} &= \frac{-n^2}{12} + \frac{n^4}{12} - nt_2 + t_2^2 - \frac{nt_1}{3} + \frac{n^3t_1}{3} + \frac{n^2t_1^2}{2} + \frac{nt_1^3}{3} + \frac{t_1^4}{12}\\
P_{123} &= \frac{n}{3} + \frac{n^2}{2} + \frac{n^3}{6} - nt_2 + \frac{nt_1}{2} + \frac{n^2t_1}{2} - t_2t_1 + \frac{nt_1^2}{2} + \frac{t_1^3}{6}.
\end{aligned} \tag{16}$$

It turns out that a more general class of rational solutions can be constructed from the τ function given by $\tau_n = W(f_n^{(1)}, f_n^{(2)}, \ldots, f_n^{(N)})$ with the $f_n^{(j)}$'s given by

$$f_n^{(j)} = \left(\frac{\partial}{\partial p_j}\right)^{m_j} \exp[\eta(p_j)] \equiv P_{m_j}(p_j)\cdot\exp[\eta(p_j)] \qquad j = 1, 2, \ldots, N, \qquad m_j \geqslant 0 \tag{17}$$

and they satisfy the equations (6) with

$$\eta(p_j) = (n + n_j)\log(1 + p_j) + p_j(t_1 + \tilde{t}_{1j}) + p_j^2(t_2 + \tilde{t}_{2j}) \tag{18}$$

where n_j, $\tilde{t}_{1j}$ and $\tilde{t}_{2j}$ are arbitrary phase constants. From (17), we have

$$P_{m_j}(p_j) = m_j! \sum_{\substack{\alpha_0,\alpha_1,\alpha_2,\geqslant 0\\ \alpha_0+\alpha_1+2\alpha_2=m_j}} \prod_{k=0}^{m_j} \frac{(\theta_k(p_j))^{\alpha_k}}{\alpha_k!} \tag{19}$$

where

$$\theta_k(p_j) = \frac{1}{k!}\frac{\partial^k}{\partial {p_j}^k}\eta(p_j). \tag{20}$$

These polynomials $P_{m_j}(p_j)$ are the generalized Schur polynomials [12, 28, 50]. Again, it should be noted that these generalized Schur polynomials and the Wronskian formed by them are also rational solutions for the DΔKP equation. But this time the entries in the determinant are arbitrary linear combinations of Schur polynomials.

We observed that all these rational solutions are the special cases of those represented by equation (5) together with (6). Hence, it is clearly demonstrated that the soliton solutions and various kinds of rational solutions discussed in this paper quite naturally arise out of the τ function derived in the framework of Sato theory.

By introducing an infinite number of time variables in the linear equations

$$\frac{\partial}{\partial t_m} f_n^{(j)} = \Delta^m f_n^{(j)} \qquad j = 1, 2, \ldots N, \qquad m = 1, 2, \ldots \tag{21}$$

it is straightforward to generalize the above ideas for obtaining the Wronskian and rational solutions for DΔKP hierarchy.

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